

Hybrid Percolation Transition in Cluster Merging Processes: Continuously Varying Exponents

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Consider growing a network, in which every new connection is made between two disconnected nodes. At least one node is chosen randomly from a subset consisting of g fraction of the entire population in the smallest clusters. Here we show that this simple strategy for improving connection exhibits a phase transition barely studied before, namely a hybrid percolation transition exhibiting the properties of both first-order and second-order phase transitions. The cluster size distribution of finite clusters at a transition point exhibits power-law behavior with a continuously varying exponent τ in the range $2 < \tau(g) \leq 2.5$. This pattern reveals a necessary condition for a hybrid transition in cluster aggregation processes, which is comparable to the power-law behavior of the avalanche size distribution arising in models with link-deleting processes in interdependent networks.

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Transport or communication systems grow by adding new connections. Often certain constraints are imposed by society and if these constraint involve global knowledge about connectivity, the transition to a percolating system can become first order, as happens for instance when suppressing the spanning cluster, when imposing a cluster size [1] or when favoring the most disconnected sites [2]. Typically this effect is accompanied by the loss of critical scaling making these abrupt transitions less predictable and thus more dangerous. We will show here, that for a specific case, namely a variant of the model introduced in Ref. [3], critical fluctuations and power-law distributions can prevail and for the first time identify a hybrid transition in explosive percolation.

Hybrid phase transitions have been observed recently in many complex network systems [4, 5]; in these transitions, the order parameter $m(t)$ exhibits behaviors of both first-order and second-order transitions simultaneously as

$$m(t) = \begin{cases} 0 & \text{for } t < t_c, \\ m_0 + r(t - t_c)^\beta & \text{for } t \geq t_c, \end{cases} \quad (1)$$

where m_0 and r are constants and β is the critical exponent of the order parameter, and t is a control parameter. Examples of such behavior include k -core percolation [6, 7], the cascading failure model on interdependent complex networks [8, 9], and the Kuramoto synchronization model with a correlation between the natural frequencies and degrees of each node on complex networks [10, 11], etc. For the models in [6–9], a critical behavior appears as nodes or links are deleted from a percolating cluster above the percolation threshold until reaching a transition point t_c . As t is decreased infinitesimally further as $1/N$ beyond t_c in finite systems, the order parameter decreases suddenly to zero and a first-order phase transition occurs. Thus, a hybrid phase transition

occurs at $t = t_c$ in the thermodynamic limit.

Next we recall discontinuous percolation transitions occurring in generalized contagion models [12–14]. Recent studies [15] of a generalized epidemic model [13] revealed that the discontinuous percolation transition turns out to be a hybrid percolation transition (HPT) represented by (1). For this case, a HPT is induced by cluster merging processes. However, the critical behavior arising in a HPT in a cluster merging process has not been yet studied at all, even though one may guess that its nature can differ from that of the HPT in link-deleting processes [6–9]. This Letter aims to understand the nature of an HPT in a cluster merging process and identify the similarities and differences in the phase transition compared with those of HPTs in link-deleting processes. Our study is based on a simple stochastic model introduced later, from which we could obtain analytic solutions for diverse properties of the critical behavior.

The model we study is defined as follows: We start with a system consisting of N isolated nodes. At each time step, one node is selected uniformly at random from the entire system and the other node is selected from a restricted set consisting of approximately g fraction of the entire nodes, as defined below. The two nodes are connected by an edge unless they are already connected. The time step t is defined as the number of edges added to the system per node, which is a control parameter. The restricted set is defined as follows: First, we rank clusters by ascending order of size at each time step. When more than one cluster of the same size exists, they are sorted randomly. At this stage, a restricted set of clusters $R(t)$ is defined as the subset consisting of a certain number of the smallest clusters (say k clusters), denoted as $R(t) \equiv \{c_1, c_2, \dots, c_k\}$. Further, k is determined as the value satisfying the inequalities, $N_{k-1}(t) < \lfloor gN \rfloor \leq N_k(t)$ for a given model parameter $g \in (0, 1]$. $N_k(t) \equiv \sum_{\ell=1}^k s_\ell(t)$,

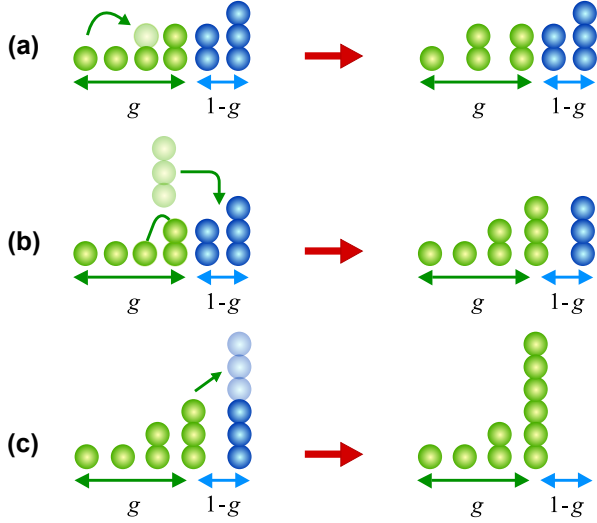


FIG. 1. (Color online) Schematic illustration of the r-ER model with $g = 0.5$ and $N = 10$. Nodes (represented by balls) in $R(t)$ are green (light gray), whereas those in $R^{(c)}(t)$ are blue (dark gray). Columns represent clusters. At each time step, two nodes are selected, one from the green balls and the other from balls of any color. (a) When the two balls are both green, if the sum of their cluster sizes is smaller than or equal to $S_R(t)$, the size of the largest cluster in $R(t)$, then $S_R(t)$ remains the same. (b) However, if the sum is larger than $S_R(t)$, then $S_R(t)$ can be increased. The cluster of two blue balls no longer belongs to $R^{(c)}(t)$ but moves to $R(t)$. The merged cluster is still in $R(t)$ because it contains a ball with ranking gN . (c) When the size of the largest cluster in the system exceeds $(1-g)N$ for the first time, which is referred to as t_g , all nodes are regarded as green balls. After this happens, the dynamics follows that of the ordinary ER model.

where $s_\ell(t)$ is the number of nodes in cluster c_ℓ . We note that the number of clusters in $R(t)$ varies with time. For later discussion, we denote the number of nodes in the set $R(t)$ as $N_R(t) \equiv N_k(t)$ and the size of the largest cluster in $R(t)$ as $S_R(t)$. This model is called a restricted Erdős and Rényi (r-ER) model, because when $g = 1$, the model is reduced to percolation in the ordinary ER model. The model is depicted schematically in Fig. 1. We remark that this r-ER model is a slightly modified version of the original model [3] in which the number of nodes in the set R is always $\lfloor gN \rfloor$, independent of time. Thus, when $N_{k-1}(t) < \lfloor gN \rfloor < N_k(t)$, some nodes in one of the largest clusters in R belong to R and the other nodes of the same cluster belong to $R^{(c)}$. However, in our modified model, all the nodes in that cluster belong to the set R . This modification enables us to solve analytically the phase transition for $t > t_c$ without changing any critical properties (see supplementary information (SI))

The r-ER model exhibits an HPT at a percolation threshold t_c , which is delayed compared with the percolation threshold of the ordinary ER model as shown in Fig. 2. This behavior is similar to that obtained in the

explosive percolation model [1, 16–18]. The order parameter is the fraction of nodes belonging to the giant cluster, denoted as $m(t)$. This order parameter begins to increase abruptly from a certain tipping point t_c^- defined in [19] (see also Fig. 3b) and reaches a finite value $m_0 < 1$ at t_c . Thus during the interval $\Delta t \equiv t_c - t_c^-$, the order parameter increases drastically. The interval Δt scales as $o(N)/N$ [3], which reduces zero in the limit $N \rightarrow \infty$. Thus, the abrupt transition becomes a discontinuous transition in the thermodynamic limit. This property is also confirmed using finite-size scaling analysis, which is presented in the SI. After t_c , the order parameter $m(t)$ increases gradually. Here we argue that in general, continuously increasing behavior of $m(t)$ for $t > t_c$ does not guarantee an HPT. One needs to check whether the order parameter increases continuously following Eq. (1), and other physical quantities follow critical behaviors.

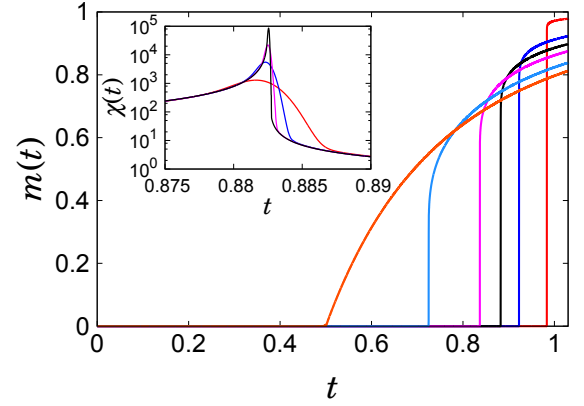


FIG. 2. (Color online) Plot of $m(t)$ vs t for different model parameter values $g = 0.2, 0.4, 0.5, 0.6, 0.8$ and 1.0 from right to left. Inset: Plot of the susceptibility $\chi(t)$ vs t for different system sizes $N/10^4 = 8, 64, 512$ and 4096 for a fixed $g = 0.5$.

We examine the cluster size distribution $n_s(t)$ for several time steps as shown in Fig. 3, where $n_s(t)$ denotes the number of clusters of size s divided by N . i) When $t \ll t_c$, $n_s(t)$ decays exponentially with respect to s . ii) When $t = t_c^-$, $n_s(t)$ exhibits power-law decay in the small-cluster region but contains a bump in the large-cluster region. iii) When $t = t_c$, at which the order parameter becomes m_0 , $n_s(t)$ exhibits power-law decay with the exponent τ for finite clusters, at an s -distance from a giant cluster positioned separately at $s = m_0N$. Note that τ depends on the model parameter g . iv) When $t > t_c$, the size distribution of finite clusters exhibits crossover behavior: it undergoes power-law decay for $s < s^*$, but exponential decay for $s > s^*$. Thus, $n_s(t) \sim s^{-\tau} e^{-s/s^*}$, where $s^* \sim (t - t_c)^{-1/\sigma}$ according to the conventional scaling theory.

We consider the evolution of the cluster size at an arbitrary time step t using the Smoluchowski equations for

the cluster size distribution $n_s(t)$.

$$\frac{dn_s(t)}{dt} = \sum_{i,j=1}^{\infty} \frac{in_i j n_j}{g} \delta_{i+j,s} - \left(1 + \frac{1}{g}\right) s n_s \quad \text{for } s < S_R, \quad (2)$$

$$\frac{dn_s(t)}{dt} = \sum_{i,j=1}^{\infty} \frac{in_i j n_j}{g} \delta_{i+j,s} - s n_s - \left(1 - \sum_{k=1}^{S_R-1} \frac{in_i}{g}\right) \quad \text{for } s = S_R, \quad (3)$$

$$\frac{dn_s(t)}{dt} = \sum_{j=1}^{\infty} \delta_{i+j,s} j n_j \sum_{i=1}^{S_R-1} \frac{in_i}{g} + \sum_{j=1}^{\infty} \delta_{S_R+j,s} j n_j \left(1 - \sum_{i=1}^{S_R-1} \frac{in_i}{g}\right) - s n_s \quad \text{for } s > S_R. \quad (4)$$

Note that the number of clusters of size S_R can be greater than one. In this case, some of them are randomly chosen to belong to $R(t)$ and the others belong to $R^{(c)}$ to satisfy the inequalities $N_{k-1}(t) < \lfloor gN \rfloor \leq N_k(t)$ presented previously. Thus, we need to consider the case $s = S_R$ separately. In the above equations, we used the approximation that the total number of nodes in the set R is $N_R \approx gN$ for $t < t_g$, where t_g is the time step at which the size of the largest cluster in the system exceeds $(1-g)N$ for the first time. Thus, g is taken as unity for $t \geq t_g$. Further, t_g is located between t_c^- and t_c . The derivations of each term for each case of s are explained in the SI. Moreover, performing direct numerical integration of the Smoluchowski equations, we successfully reproduce the behaviors of $m(t)$ in Fig.2 and n_s in Fig.3, which are shown in the SI.

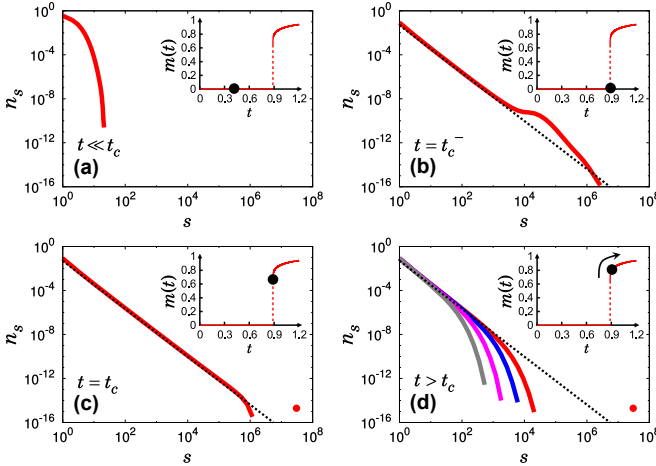


FIG. 3. (Color online) Plot of $n_s(t)$ vs s at several time steps, (a) $t \ll t_c$, (b) $t = t_c^-$, (c) $t = t_c$, and (d) $t > t_c$. Inset in each panel indicates how the order parameter $m(t)$ behaves at each time.

We focus on the rate equation for $n_1(t)$ as follows: At early initial time steps, monomers are abundant and can be in both R and $R^{(c)}$. However, as time passes, their number decreases and at a certain time step denoted as

t_1 , all the monomers are in the set R . The rate equation for monomers is written separately as $\frac{dn_1}{dt} = -n_1 - 1$ for $t < t_1$ using Eq.(3), and $\frac{dn_1}{dt} = -\left(1 + \frac{1}{g}\right) n_1(t)$ for $t > t_1$ using Eq.(2). Then

$$n_1(t) = \begin{cases} 2e^{-t} - 1 & \text{if } t < t_1, \\ g\left(\frac{g+1}{2}\right)^{-(1+\frac{1}{g})t} e^{-(1+\frac{1}{g})t} & \text{if } t > t_1, \end{cases} \quad (5)$$

and t_1 is obtained as $t_1 = \ln\left(\frac{2}{g+1}\right)$.

Next, we use the fact that $n_s(t_c)$ follows a power law, i.e., $n_s(t_c) = a_0 s^{-\tau}$ for $1 \leq s \leq S_0$, where S_0 is the size of the largest cluster among the finite clusters and a_0 can be determined in terms of g and t_c from Eq. (5). We can also use the relation $\sum_{s=1}^{\infty} n_s(t_c) = 1 - t_c$, because up to t_c , links are likely to be added between clusters, which is checked numerically in the SI. We also confirm that $d \sum_{s=1}^{\infty} n_s(t)/dt = -1$ from the Smoluchowski equations (2-4). Moreover, the relation $\sum_{s=1}^{S_0} s n_s(t_c) \approx \sum_{s=1}^{\infty} s n_s(t_c) = 1 - m_0$ can be used in the limit $N \rightarrow \infty$. Taking account of those facts, we obtain the self-consistency equation for the exponent τ as

$$\frac{\zeta(\tau)}{\zeta(\tau-1)} = \frac{1}{1-m_0} \left\{ 1 - \ln \left(\left[\frac{g\zeta(\tau-1)}{1-m_0} \right]^{\frac{g}{1+g}} \left(\frac{2}{g+1} \right) \right) \right\}. \quad (6)$$

To obtain τ , one needs m_0 values for given g values. We use the numerically estimated values of m_0 in Table I. The obtained values denoted as τ range between $2 < \tau(g) \leq 2.5$, as listed in Table I. We note that as g decreases, τ becomes smaller, and the jump in the order parameter becomes larger. This result implies that the growth of large clusters is more strongly suppressed at smaller g .

To obtain the exponent σ analytically, we take into account two facts: First, the dynamics in regime iv) is of the ER type: Because $m_0 > 1 - g$, the restricted set covers the entire system. Next we take t_c as an *ad hoc* time origin, and the cluster size distribution at t_c is used as an initial condition of the ER dynamics. Under this transformation, the solution for the evolution of the ER

model remains the same [20]. Then using the solution for the cluster size distribution of the ER model [21], we obtain that $sn_s(\hat{t}) = s^{1-\tau} f(s\hat{t}^{1/\sigma})$, where $f(x)$ is a scaling function and $\hat{t} \equiv t - t_c$ for $t > t_c$. Using the property that $f(x)$ is an analytic function [22], we obtain $\sigma = 1$ independent of g . The detailed derivation is presented in the SI.

The order parameter $m(t)$ increases continuously with time when $t > t_c$. To study the criticality for $t > t_c$, we again take the transition point t_c as an *ad hoc* time origin. Next, we use the formalism for the ordinary ER model with an arbitrary initial cluster size distribution as presented in [20–22]. For instance, the order parameter can be obtained via the self-consistency equation, $m(\hat{t}) = H(2tm(\hat{t}))$, where $H(\mu) \equiv 1 - \sum_s sn_s(0)e^{-\mu s}$ [20, 21]. Using $n_s(\hat{t} = 0) = a_0 s^{-\tau}$ with $a_0 = (1 - m_0)/\zeta(\tau - 1)$, we obtain that $m(\hat{t} = t - t_c) = m_0 + r(t - t_c)^{\tau-2}$, where $r = -\frac{\Gamma(2-\tau)}{\zeta(\tau-1)}(1-m_0)(2m_0)^{\tau-2}$. This gives $\beta = \tau - 2$. The detailed derivation of the exponent β is presented in the SI. Note that we used $m(\hat{t}) = m_0 + r\hat{t}^\beta$ instead of $m(\hat{t}) \sim \hat{t}^\beta$ to obtain the above result. If we had used $m(\hat{t}) \sim \hat{t}^\beta$, relevant to the case $m_0 = 0$, we would have obtained $\beta = (\tau - 2)/(3 - \tau)$. For the ER model, $\tau_{ER} = 5/2$, and $\beta_{ER} = 1$. Accordingly, we say that the discontinuity of the order parameter $m_0 \neq 0$ at t_c changes the criticality for $t > t_c$.

The susceptibility $\chi^+ \equiv \sum_{s=1}' s^2 n_s(t)$ for $t > t_c$, where the prime indicates the exclusion of an infinite-size cluster, can be obtained using $\chi^+ = H'(2\hat{t}m(\hat{t}))/[1 - 2\hat{t}H'(2\hat{t}m(\hat{t}))]$ with $\hat{t} = t - t_c$, where the prime in $H(\mu)$ indicates the derivative with respect to its argument μ . When $2 < \tau < 3$, $H'(\mu)$ has a non-integer singularity of $\mu^{\tau-3}$. Plugging $\mu = 2\hat{t}m_0$ into the formula for $\chi^+(t)$, we obtain that $\chi^+(t) \sim (t - t_c)^{-\gamma}$ with $\gamma = 3 - \tau$. Note that the denominator is finite at $\hat{t} = 0$.

Obtaining the analytical result of the susceptibility $\chi^-(t) \equiv \sum_{s=1}^\infty s^2 n_s(t)$ for $t < t_c$ is intriguing. Using the Smoluchowski equation for the r-ER model, we obtain the following relation $\partial\chi^-(t)/\partial t = 2\chi^-(t)\chi_R(t)$, where $\chi_R \equiv (1/g)\sum_{s=1}^{S_R} s^2 n_s(t)$ and we used the approximation $\sum_{s=1}^{S_R} sn_s \approx g$ which is valid near t_c^- . If we assume that $\chi^-(t)$ diverges algebraically as $t \rightarrow t_c$, i.e., $\chi^- \sim (t_c - t)^{-\gamma'}$, which is confirmed numerically, then we could obtain $\chi_R = (\gamma'/2)(t_c - t)^{-1}$. However, the exponent γ' was not analytically determined. In fact, χ_R is another form of the susceptibility defined via the correlation function [23] as $\chi_c \equiv (1/N)\sum_{i,j=1}^N C(i,j)$, where $C(i,j)$ is the probability that two nodes i and j belong to the same finite cluster. Thus, $C(i,j)$ can be obtained as the probability that two nodes i and j are selected from the same cluster. Then, $\chi_c = N\sum_{\alpha \in R} (s_\alpha/N)(s_\alpha/gN)$, where α is the index of cluster. This formula reduces to χ_R using $N_R/N \approx \sum_{s=1}^{S_R} sn_s \approx g$ near t_c^- .

A critical behavior at t_c appears in the form of the power-law-type size distribution of finite clusters, which

is necessary to generate an HPT, with exponent τ ranging between $2 < \tau \leq 2.5$. This pattern is analogous to the power-law behavior of the avalanche size distribution in the cascading failure model [8].

The r-ER model is a simple model exhibiting an HPT in a cluster merging process. As the classical ER model has served as a basic model for understanding the evolution of social networks, the r-ER model should be similarly useful but under a certain constraint that the least connected members are preferred to connect to others, for instance, as in the merging of the lowest financial companies under government control in financial crisis. Moreover, the theoretical framework obtained from the r-ER model can be used for understanding other HPTs in cluster merging processes, for instance, in generalized epidemic models [13, 15] and synchronization models [10, 11]. Our preliminary results suggest that indeed HPTs in cluster merging processes could be found from those models.

We have shown here, that in a model, in which the most disconnected agents are preferentially connected, spanning occurs abruptly, while astonishingly critical fluctuations and power-law distributions prevail after the transition within the critical region. These post-transition critical mergers of rather big clusters into a rather meager spanning one represent a rather transparent geometrical interpretation of the recently discovered hybrid phase transitions.

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TABLE I. For a given model parameter g , percolation threshold (t_c), discontinuity of the order parameter at t_c (m_0), the exponent of the cluster size distribution at t_c obtained from Eq. (6) using the mean value m_0 (τ^*), numerically obtained (τ), the exponent ($1/\sigma$), the exponent (β), the exponent (γ') for $t \rightarrow t_c^-$, and (γ) for $t \rightarrow t_c$. All simulation results are obtained from several different system sizes up to $N = 4 \times 10^7$. The estimated values of the critical exponents and error bars are confirmed to be independent of the system sizes. The details of how to measure them are presented in the SI.

g	t_c	m_0	τ^*	τ	$1/\sigma$	β	γ'	γ
0.1	0.9984 ± 0.0002	0.98 ± 0.05	2.06	2.03 ± 0.04	1.00 ± 0.05	0.04 ± 0.05	1.00 ± 0.02	0.97 ± 0.05
0.2	0.9836 ± 0.0002	0.90 ± 0.05	2.13	2.08 ± 0.04	1.01 ± 0.05	0.09 ± 0.05	1.03 ± 0.02	0.91 ± 0.05
0.3	0.9570 ± 0.0002	0.81 ± 0.05	2.18	2.12 ± 0.04	1.01 ± 0.05	0.13 ± 0.05	1.06 ± 0.03	0.88 ± 0.05
0.4	0.9227 ± 0.0002	0.72 ± 0.05	2.23	2.16 ± 0.04	1.02 ± 0.05	0.17 ± 0.04	1.08 ± 0.03	0.85 ± 0.05
0.5	0.8826 ± 0.0002	0.62 ± 0.05	2.26	2.18 ± 0.04	1.04 ± 0.05	0.21 ± 0.05	1.10 ± 0.03	0.83 ± 0.05
0.6	0.8370 ± 0.0002	0.52 ± 0.06	2.29	2.20 ± 0.04	1.05 ± 0.05	0.23 ± 0.04	1.11 ± 0.04	0.82 ± 0.05
0.7	0.7852 ± 0.0003	0.41 ± 0.05	2.32	2.22 ± 0.04	1.08 ± 0.05	0.28 ± 0.05	1.13 ± 0.05	0.81 ± 0.05
0.8	0.7250 ± 0.0003	0.29 ± 0.05	2.35	2.25 ± 0.04	1.12 ± 0.05	0.32 ± 0.05	1.15 ± 0.05	0.81 ± 0.05
0.9	0.650 ± 0.002	0.15 ± 0.05	2.38	2.28 ± 0.04	1.20 ± 0.10	0.39 ± 0.08	1.15 ± 0.05	0.80 ± 0.05

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